Finding a marked node on any graph by continuous-time quantum walk

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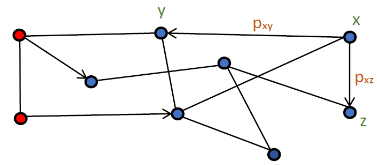
Classical random walk on a graph



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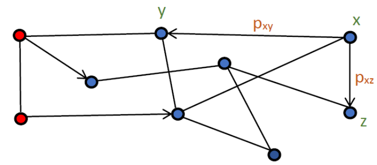

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- Described by a $n \times n$ stochastic matrix P such that its $(x, y)^{\mathrm{th}}$ entry is p_{xy}.
- If the row-vector v_0 is the initial state of the walker, after t-steps: $v_t = v_0 P^t$.

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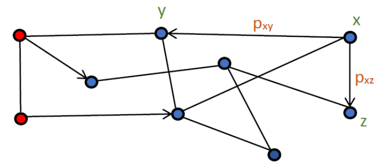

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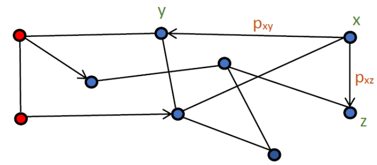

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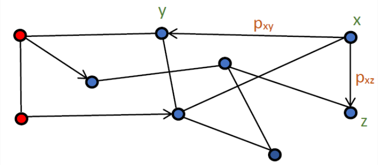

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  - Eigenvalues of P lie between -1 and 1.
  - π is unique.

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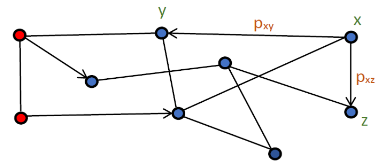

Set of marked nodes:
$M \subseteq X$.

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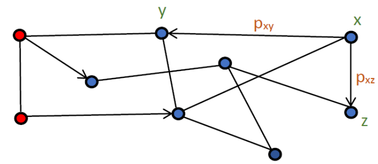

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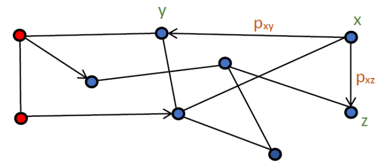

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Spatial search (classical)

1. Sample $x \in X$ from π.
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3. If $x \in M$, output x
4. Otherwise update x according to P and go to step 2.

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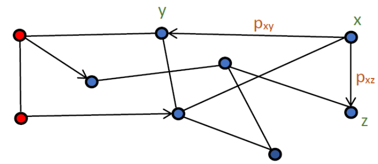

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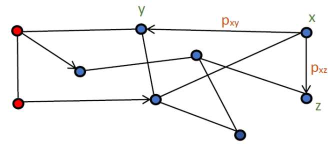
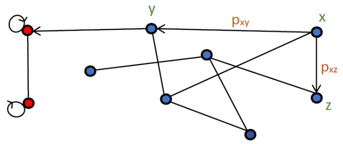
Hitting time of P with respect to M is the expected number of times step 4 is executed.

Classical Hitting time

Spatial search stops when $x \in M \implies$ Walk on an absorbing Markov chain P'





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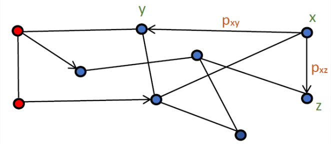

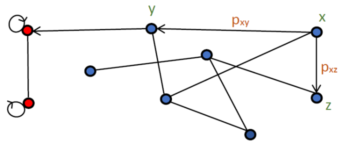

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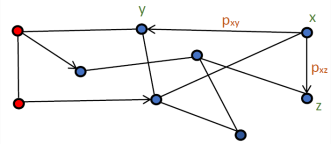

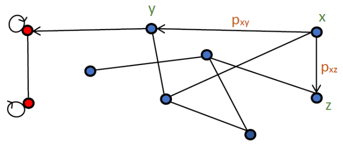

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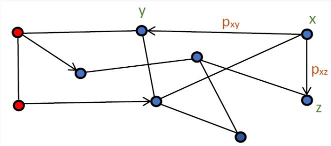

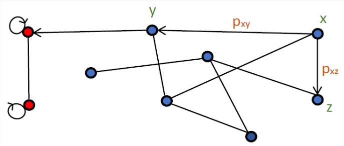

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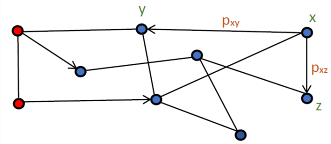

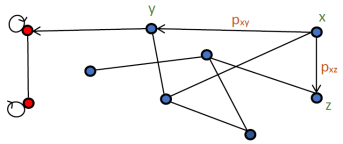

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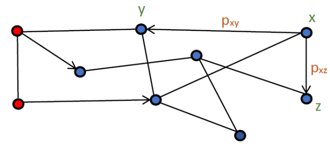
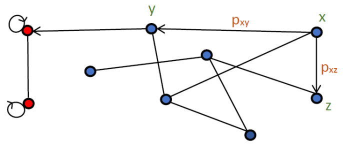
Framework

Let $U = X \backslash M$.





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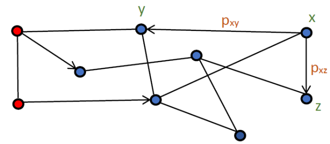

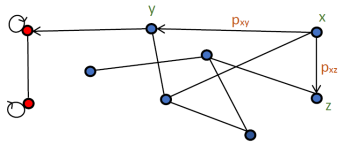

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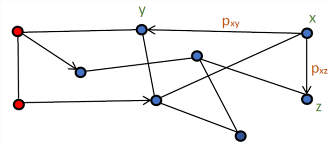

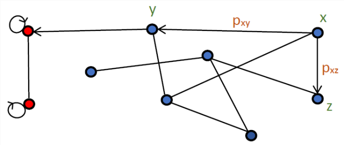

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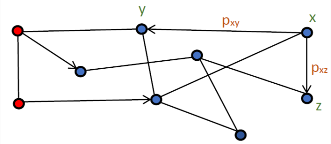

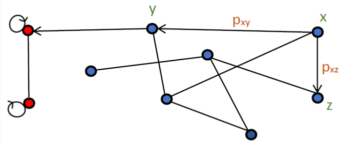

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  - Spectral gap: $\Delta(s) = 1 - \lambda_{n-1}(s)$.

The known and the unknown

Complexity of spatial search by DTQW

For any *ergodic*, *reversible* Markov chain P with a set of M marked nodes: $\mathcal{O}\left(\sqrt{HT^{+}(P,M)}\right)$.
[Krovi, Magniez, Ozols, and Roland 2014]

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Complexity of spatial search by CTQW

No such general result known.

The algorithm by Childs and Goldstone has been applied to certain specific graphs such as d-dimensional lattices, hypercubes and others [Childs and Goldstone 2004 and several subsequent papers].

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- State general conditions for the optimality of the Childs and Goldstone algorithm on any ergodic, reversible Markov chain.

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- For any ergodic, reversible Markov chain provide a spatial search algorithm by CTQW that has a running time of $\Theta(\sqrt{HT^{+}(P,M)})$.
- State general conditions for the optimality of the Childs and Goldstone algorithm on any ergodic, reversible Markov chain.
- Compare the running time of our algorithm with the Childs and Goldstone algorithm.

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Task

Convert a graph (or a Markov chain) to a Hamiltonian.
We will use the formalism of Somma and Ortiz [Somma and Ortiz 2010].
Also used to develop an adiabatic version of quantum spatial search [Krovi, Ozols and Roland 2010].

Search Hamiltonian

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$|v_n(s)\rangle = \sqrt{\pi(s)^T}$.

$\mathcal{H}$: n-dimensional Hilbert space whose basis states are labelled by the vertices of P.

Consider a unitary $V(s) \in \mathcal{H} \times \mathcal{H}$:

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If $S\,|x,y\rangle = |y,x\rangle$, then observe that

$$\langle x,0|V^\dagger(s)SV(s)|y,0\rangle = \sqrt{p_{xy}(s)p_{yx}(s)} = D_{xy}(s).$$

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$$V(s) |x,0\rangle = \sum_{y \in X} \sqrt{p_{xy}(s)} |x,y\rangle .$$

If $S|x,y\rangle = |y,x\rangle$, then observe that

$$\langle x,0| V^\dagger(s) S V(s) |y,0\rangle = \sqrt{p_{xy}(s)p_{yx}(s)} = D_{xy}(s).$$

If $\Pi_0 = I \otimes |0\rangle\langle 0|$,

$$H(s) = i[V^\dagger(s) S V(s), \Pi_0].$$

Spectrum of $H(s)$

- $|v_n(s), 0\rangle$ is an eigenstate of $H(s)$ with eigenvalue 0.
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For $s = s^* = 1 - p_M/(1-p_M)$, $|v_n(s^*)\rangle = \frac{|U\rangle + |M\rangle}{\sqrt{2}}$.

$$p_M = \sum_{x \in M} \pi_x$$

$$|U\rangle = \frac{1}{\sqrt{1-p_M}} \sum_{x \notin M} \sqrt{\pi_x}\, |x\rangle$$

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Algorithm

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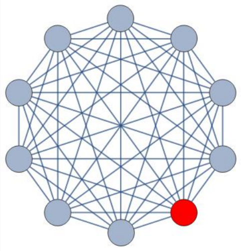
Complete graph:

$$A_{ij} = 1, i \neq j$$

Same as analog Grover

$$\begin{aligned} H &= -|w\rangle\langle w| - |s\rangle\langle s| \\ &= -|w\rangle\langle w| - \frac{1}{n}A \end{aligned}$$

Optimal! $T = O(\sqrt{n})$



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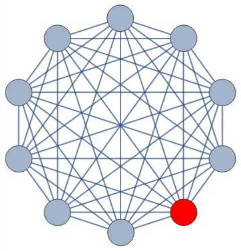

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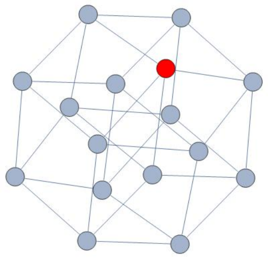

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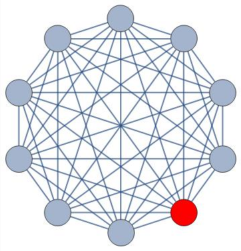

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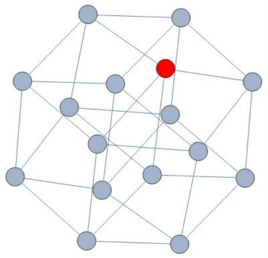


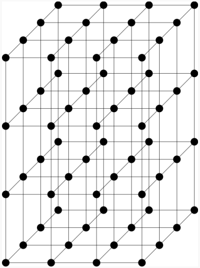
Square Lattices:

$d \leq 3$
No significant speed-up

$d = 4$

$$T = O(\sqrt{n} \log n)$$

$d > 4$
Optimal! $T = O(\sqrt{n})$



Sufficient condition for the optimality of $\mathcal{CG}$ algorithm

Let H_G be a Hamiltonian with eigenvalues

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Apply $\mathcal{O}(1/\nu)$ rounds of amplitude amplification: $T_{search} = \mathcal{O}\left(\frac{1}{\nu^2\sqrt{\epsilon}}\right)$.

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- Algorithm is optimal for any symmetric P having $\nu = \Theta(1)$.

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  - T_{search} is sub-optimal even for state-transitive P!

Example: Movement of rook on a rectangular chessboard

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to rotate from $|v_n, 0\rangle$ to the state

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Conclusions

- Spatial search algorithm by CTQW that runs in $\Theta(\sqrt{HT^+(P,M)})$ time on any ergodic, reversible Markov chain.
- Provided general conditions for the optimality of the spatial search algorithm by Childs and Goldstone
  - Applicable to only state-transitive Markov chains.
  - Sub-optimal.
- Improved the Childs and Goldstone algorithm to be applicable to any ergodic, reversible Markov chain.

Open questions:

Extended hitting time vs hitting time

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Future work:

Open questions:

Extended hitting time vs hitting time

Future work:

Quantum algorithms for preparing the stationary state of an ergodic, reversible Markov chain.

Thank you for your attention!

For more details see:

S. Chakraborty, L. Novo and J. Roland, Finding a marked node on any graph by continuous time quantum walk, arXiv:1807.05957 (2018).