Applying quantum algorithms to constraint satisfaction problems

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Introduction

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- Many-body localisation [Childs et al '17]
- Other problems in quantum chemistry and condensed-matter physics, e.g. [Babbush et al '18]
- Integer factorisation [Kutin '06]

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But what if we don't care about cryptography or simulation of quantum systems?

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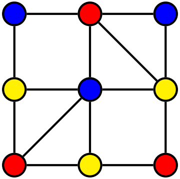
2. It should solve a problem that (many) people care about in a reasonable time (e.g. < 1 day).
3. We should compare it against the best classical algorithms running on real hardware.

Two applications: k-SAT and graph k-colouring



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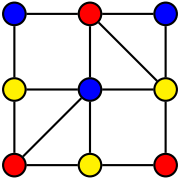

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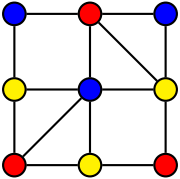

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Each problem is NP-complete and has a huge number of direct applications:

- SAT: verification of electronic circuits; planning; computer-aided mathematical proofs; ...
- Colouring: register allocation; scheduling; frequency assignment problems; ...

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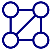
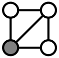
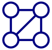
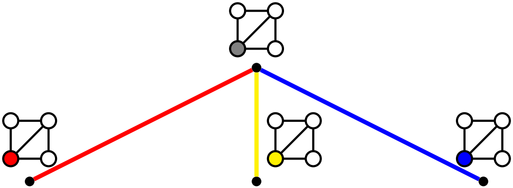
5. We compared against the likely performance of leading classical algorithms (Maple_LCM_Dist and DSATUR).

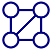
Colouring by backtracking

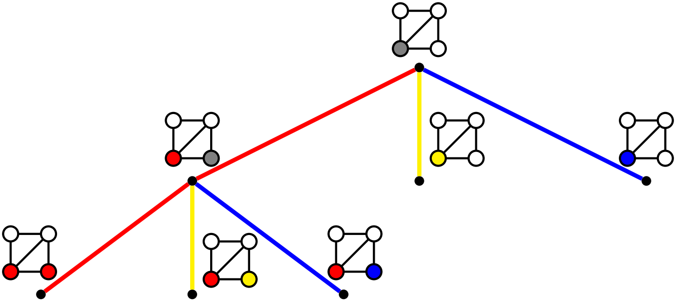
Colouring  by backtracking

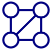


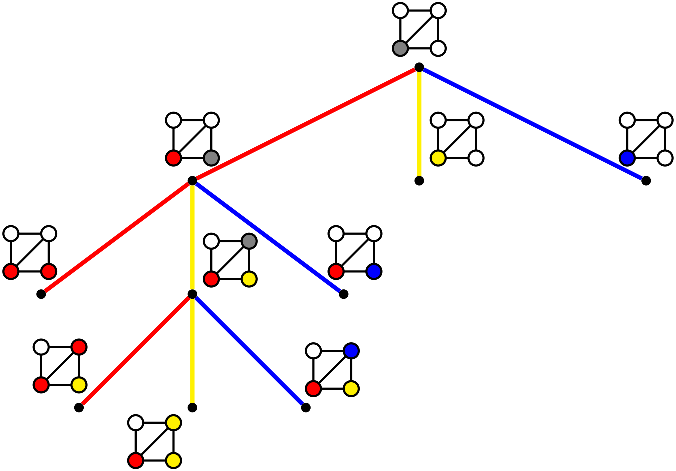
Colouring  by backtracking

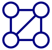


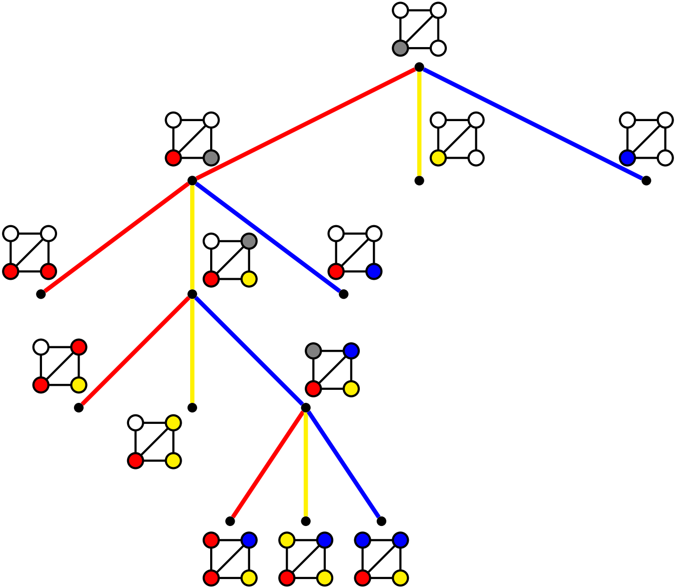
Colouring  by backtracking



Colouring  by backtracking



Colouring  by backtracking



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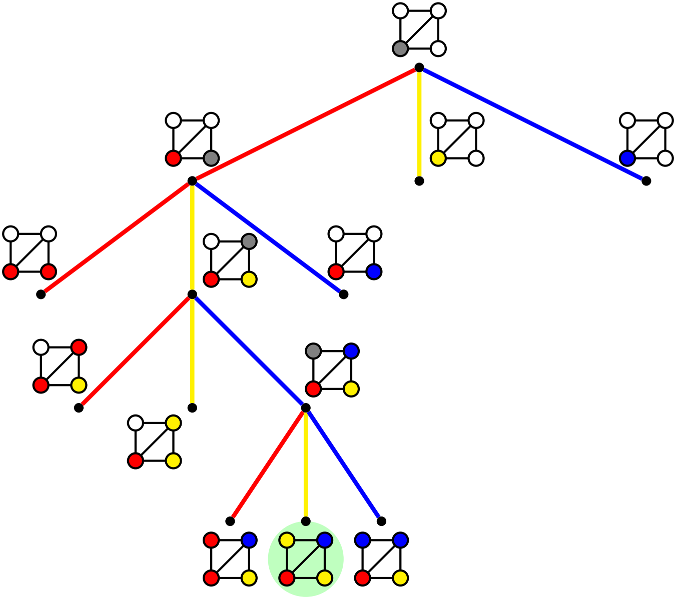

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We then convert this to real-world runtimes based on various regimes corresponding to different parameters for quantum-computing hardware:

Parameter	Realistic	Plausible	Optimistic
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"Realistic" is (approximately) achievable today; other two columns represent order-of-magnitude improvements.

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- We assume that Clifford gates are free and T/Toffoli gates can be implemented in time equal to the cost of one measurement.

- This is justified by the use of time-optimal techniques [Fowler '12, Eastin '13, ...] to prepare magic states offline and then inject them at the cost of 1 measurement.

Summary of results: good and bad news

- In the most optimistic hardware parameter regime, we could see speedup factors of $> 10^5$ (compared with a standard desktop PC) for k-SAT (via Grover's algorithm) and $> 10^4$ for graph colouring (via backtracking).

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- If we additionally take into account the cost of classical error-correction processing, this speedup essentially disappears.
- The number of physical qubits used is very large (e.g. $> 10^{12}$), almost all of which are used for fault-tolerance.
- This strongly motivates the design of improved fault-tolerance techniques!

Summary of results (1)

	Realistic	Plausible	Optimistic
Max n	65	72	78
T-depth	1.46×10^{12}	1.65×10^{13}	1.32×10^{14}
Toffoli count	4.41×10^{17}	5.52×10^{18}	4.79×10^{19}
Factory qubits	3.14×10^{13}	5.15×10^{12}	1.38×10^{12}
Speedup factor	1.62×10^{3}	1.73×10^{4}	1.83×10^{5}

Table : Likely speedup factors for 14-SAT via Grover's algorithm achievable in different regimes. Relative to an Intel Core i7-4790S CPU operating at 3.20GHz.

Summary of results (2)

	Realistic	Plausible	Optimistic
Max n	55	63	72
T-depth	1.63×10^{12}	1.43×10^{13}	1.63×10^{14}
T/Toffoli count	4.72×10^{18}	4.72×10^{19}	6.16×10^{20}
Factory qubits	3.85×10^{14}	5.03×10^{13}	2.17×10^{13}
Speedup factor	1.50×10^{1}	3.92×10^{2}	1.16×10^{4}

Table : Likely speedup factors for 12-SAT via backtracking achievable in different regimes.

Summary of results (3)

	Realistic	Plausible	Optimistic
Max n	113	128	144
T-depth	1.70×10^{12}	1.53×10^{13}	1.62×10^{14}
T/Toffoli count	8.51×10^{17}	1.02×10^{19}	1.28×10^{20}
Factory qubits	6.50×10^{13}	9.54×10^{12}	3.69×10^{12}
Speedup factor	7.25×10^{0}	5.17×10^{2}	4.16×10^{4}

Table : Likely speedup factors for graph colouring via backtracking achievable in different regimes.

Cost of classical processing

N	Realistic	Plausible	Optimistic
10^{12}	4.17×10^{7}	4.30×10^{4}	9.15×10^{-1}
10^{16}	2.29×10^{12}	7.76×10^{8}	2.23×10^{4}
10^{20}	3.10×10^{16}	3.07×10^{13}	3.28×10^{8}

Table : Classical processing required to implement N Toffoli gates under different regimes, based on extrapolation of runtimes reported by [Delfosse and Nickerson '17].

- Cost measured in processor-days (where type of processor is CPU, GPU and ASIC respectively in realistic, plausible and optimistic regimes).
- Assumes that the speedup offered by GPUs and ASICs over CPUs is a factor of 100 and 10^6 respectively.

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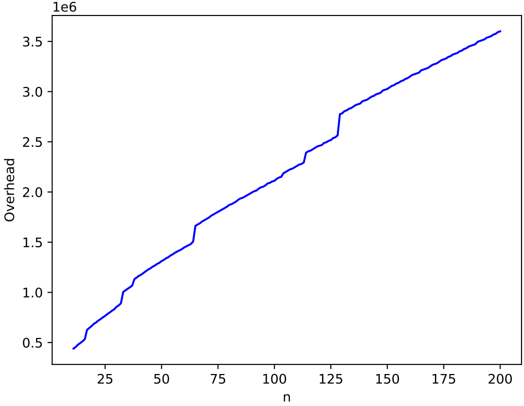

Figure : The runtime (circuit depth) of the quantum algorithm for backtracking is of the form $f(n,k)\sqrt{T}$, where T is the number of nodes in the backtracking tree. Figure illustrates scaling of $f(n,k)$ with n when k is chosen to be the expected chromatic number of a random graph.

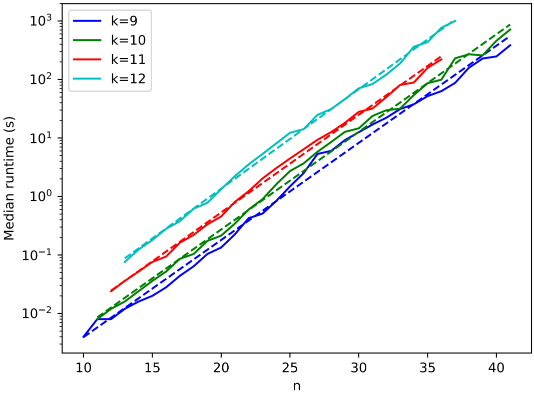

Figure : Runtime of the Maple_LCM_Dist SAT solver on random k-SAT instances with n variables and $\approx \alpha_k n$ clauses, where α_k is the satisfiability threshold. Solid line represents the median of at least 100 runs, in CPU-seconds. Dashed lines are linear least-squares fits.

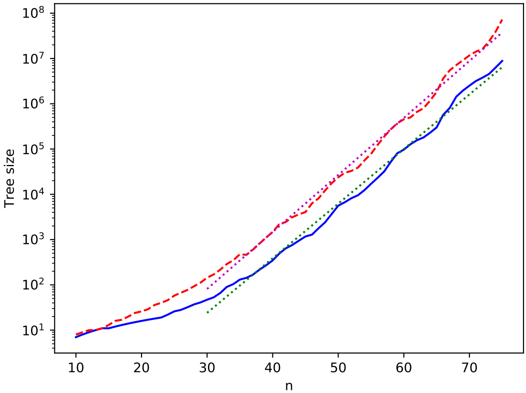

Figure : Number of nodes in the DSATUR k-colourability backtracking tree. Median (solid) and 90th percentile (dashed) over 1000 random graphs for each $n \in \{10, \ldots, 75\}$. Dotted lines are least-squares fits for the range $n \geqslant 30$.

Obtaining a quantum speedup

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- First find a large clique and colour it;
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We can accelerate DSATUR using a quantum algorithm:

Theorem (informal) [AM '15]

Let T be the number of nodes in the backtracking tree. Then there is a bounded-error quantum algorithm which runs in time $O(\sqrt{T}\,\mathrm{poly}(n))$ time and outputs whether or not an n-vertex graph is k-colourable.

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- The algorithm applies phase estimation with precision $O(1/\sqrt{Tn})$ to a quantum walk in the backtracking tree.
- The quantum walk alternates two operations, each corresponding to Grover-style diffusion D among nodes and their neighbours in the tree.
- So the overall time taken by the algorithm is

$$C_P \times \sqrt{Tn} \times 2 \times T_D$$

where $C_P \leqslant 4$ is the constant from phase estimation.

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We can achieve $T_D \leqslant 3200$ for colouring a $\leqslant$136-vertex graph.